Exponential stability of time-delay systems

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Abstract — Time delays are frequently encountered in various areas of science and engineering, including physical and chemical processes, economics, engineering, communication networks and biological systems. The existence of time delays is often a main cause of oscillations, instability and poor performance of the system. During the past decades, the stability analysis of TDS has received considerable attention from researchers, see, e.g. [1, 2] and the references therein. On the other hand, in many practical control systems, the system response is required to be as fast as possible. As a result, it is important for designers to be able to estimate the convergence rate of the system. For continuous-time systems with time varying delay, several stability analysis schemes have been proposed for deriving the exponential stability conditions; see, e.g. [3,4,5] and the references therein.

Keywords — time-delay system, stability control, Lyapunov function

I. INTRODUCTION

Along with many advantages of digital control, including cost-effectiveness and high flexibility of embedded systems, the problem of stability analysis for discrete time systems with delay has received considerable attention; see, e.g. [6] and reference therein.

In [7], a delay-dependent stability condition for discrete-time systems with time varying delay was derived by using Moon’s inequality, which depends on the minimum and maximum delay bounds. Then, further results were later reported in [8], where a set of augmented LKFs was constructed to use in conjunction with a bounding technique. Another delay-dependent stability criteria of linear continuous/discrete systems with time-varying delay were developed in [9] by using a piecewise analysis method (PAM). Based on the combination of a Lyapunov functional and the delay-partitioning approach, some stability conditions were proposed in [10], where the results were compared with those obtained by using output feedback stabilization in [11]. Improved delay-dependent stabilization criteria were reported in [12] by using a piecewise LKF and a finite sum inequality. By using the model transformation approach, another stability criteria was proposed in terms of linear matrix inequalities [13]. Based on the integral quadratic constraint (IQC) and assumption of bounded interval time-varying delay, a set of new stability criteria was presented in [14]. Recently, some improved results were reported in [13].

However, it should be noted that not much attention has been paid to discrete time systems. Moreover, the conservatism of stability conditions for linear systems with time-varying delay can be relaxed with the combination of existing LKFs, which include quadratic and double summation terms, with some free-weighting matrices. However, the use of FWMs may increase the computational complexity due to an increase in the number of decision variables. As a consequence, it is worth finding a more effective method to ultimately improve stability criteria of these systems that can be obtained in a computationally-effective manner. These together have been the motivation in the current work.

In this chapter, we consider the problem of exponential stability of discrete-time systems with interval time-varying delay. Here, without using any FWMs, we introduce a new set of LKFs containing an augmented vector and some triple summation terms. To enhance the feasible region of stability conditions, the reciprocally convex approach is used to evaluate the double summation terms in the derivative of the proposed LKFs. As a result, improved results on exponential stability are obtained, in comparison with existing stability conditions in the literature. Numerical examples are provided to illustrate the effectiveness of the proposed approach.

II. PROBLEM STATEMENT AND PRELIMINARIES

Consider the following linear discrete-time system

\[ x(k+1) = Ax(k) + Adx(k - τ(k)), \quad k \in \mathbb{Z}^+, \]

\[ x(k) = \varphi(k), \quad k \in \mathbb{Z}[-\tau_M,0], \]

where \( x(k) \in \mathbb{R}^n \) is the system state and \( A, Ad \in \mathbb{R}^{n \times n} \) are constant matrices. The time-varying delay \( τ(k) \) is assumed to belong to a given interval

\[ 0 < \tau_m \leq \tau(k) \leq \tau_M, \forall k \in \mathbb{Z}^+, \]

where \( \tau_m < \tau_M \) are positive constants representing the minimum and maximum delays respectively and \( \varphi(k), k \in \mathbb{Z}[-\tau_M,0], \) is the initial string for system.
It should be noted that system (3.1) is very popular in the literature and it is extensively studied as the time-varying delay \( \tau(k) \) is frequently encountered in many engineering systems such as networked control systems, chemical process and long transmission lines in pneumatic systems. A typical system containing time delays is the networked control system where the delays induced by the network transmission (either from sensor to controller or from controller to actuator) are actually time-varying.

The aim here is to derive new delay-dependent conditions such that system is exponential stable with the maximum allowable bound for the time delay. The following definition and lemmas are first introduced.

**Definition**

System is said to be exponentially stable if there exist positive constants \( \alpha > 1 \) and \( N > 1 \) such that all solutions \( x(k,\varphi) \) of system (3.1) satisfy

\[
\| x(k, \varphi) \| \leq N \| \varphi \| \alpha^{-k}, \forall k \in \mathbb{Z}^+
\]

where \( \alpha \) is the exponential decay rate of system

and \( \| \varphi \| = \max \{ \| \varphi(k) \| : k \in \{-\tau,0\} \} \).

For the Lyapunov-Krasovskii method, the construction of LKFs plays a crucial role in deriving the less conservative delay-dependent stability conditions. However, the estimation of the double summation terms in the different LKFs is always a challenging problem. Thus, the following lemma is used frequently.

**Lemma 1**

Let \( P \) be a symmetric positive-definite matrix and \( \tau_1, \tau_2 \in \mathbb{Z}, 0 < \tau_1 < \tau_2 \). Then for any \( r > 1 \) and a vector function \( x(k), k \in \mathbb{Z} \), the following inequality holds

\[
\sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix} \begin{bmatrix}
    x(k+u) \\
    x(k+u) \\
    x(k+u)
\end{bmatrix} \leq 
\]

\[
\sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    r^{\tau_1} & 0 & 0 \\
    0 & r^{\tau_1} & 0 \\
    0 & 0 & r^{\tau_1}
\end{bmatrix} \begin{bmatrix}
    x(k+u) \\
    x(k+u) \\
    x(k+u)
\end{bmatrix}
\]

where

\[
r_0 = \frac{1-r}{r^{\tau_1} - r^{\tau_1+1}}, r_0 = \frac{1}{r^{\tau_1+1} - 1 - r}
\]

**Proof.** By using the Schur complement, we have the following inequality for any \( r > 1 \) and \( s \in \mathbb{Z} \)

\[
\begin{align*}
    &\begin{bmatrix}
        r^{\tau_1} & 0 & 0 \\
        0 & r^{\tau_1} & 0 \\
        0 & 0 & r^{\tau_1}
    \end{bmatrix} \begin{bmatrix}
        x(k+u) \\
        x(k+u) \\
        x(k+u)
    \end{bmatrix} \\
    &\begin{bmatrix}
        x(k+u) \\
        x(k+u) \\
        x(k+u)
    \end{bmatrix} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix} \\
    &\begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix} \geq 0
\end{align*}
\]

By using Schur complement again, it follows that

\[
\sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    r^{\tau_1} & 0 & 0 \\
    0 & r^{\tau_1} & 0 \\
    0 & 0 & r^{\tau_1}
\end{bmatrix} \begin{bmatrix}
    x(k+u) \\
    x(k+u) \\
    x(k+u)
\end{bmatrix} \leq \sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix}
\]

Therefore, the following inequality can be obtained as

\[
\sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    r^{\tau_1} & 0 & 0 \\
    0 & r^{\tau_1} & 0 \\
    0 & 0 & r^{\tau_1}
\end{bmatrix} \begin{bmatrix}
    x(k+u) \\
    x(k+u) \\
    x(k+u)
\end{bmatrix} \geq \sum_{s=-\tau_2}^{-\tau_1} \sum_{u=-\tau_1-1}^{-1} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix} \begin{bmatrix}
    s+\tau_1 & 0 & 0 \\
    0 & s+\tau_1 & 0 \\
    0 & 0 & s+\tau_1
\end{bmatrix}
\]

The proof is completed.

The reciprocity convex combination lemma provided in [105] is used in this chapter. This inequality is reformulated as follows:

**Lemma 2**

For a given scalar \( \beta \in (0,1) \), an \( n \times n \) matrix \( R > 0 \) and two vectors \( \eta_1, \eta_2 \in \mathbb{R}^n \), define function \( \Theta(\beta, R) \) as

\[
\Theta(\beta, R) = \frac{\eta_1^T R \eta_1}{\beta} + \frac{1}{1-\beta} \eta_2^T R \eta_2
\]

If there is a matrix \( X \in \mathbb{R}^{n \times n} \) such that

\[
\begin{bmatrix}
    X & \eta_1 \\
    \eta_2 & R
\end{bmatrix} \geq 0,
\]

then the following inequality holds

\[
\min_{\beta \in [0,1]} \Theta(\beta, R) \geq \frac{1}{\beta} \eta_1^T R \eta_1 + \frac{1}{1-\beta} \eta_2^T R \eta_2.
\]

**Lemma 3**

Let \( V(k) \) be a Lyapunov functional, if there exist scalars \( \gamma_1 > 0, \gamma_2 > 0 \) and \( r > 1 \) such that

\[
\gamma_1 \| x(k) \| ^2 \leq V(k) \leq \gamma_2 \| x(k) \| ^2,
\]

\[
\Delta V(k) + (1-r^{-1}) V(k) \leq 0, \quad k \in \mathbb{Z}^+
\]

then every solution \( x(k,\varphi) \) of system satisfies the following estimation
\[ \|x(k, \phi)\| \leq \sqrt{\frac{72}{\tau_1}} \|\phi\| \alpha^{-k}, \quad k \in \mathbb{Z}^+ \]

where the Lyapunov factor and exponential decay rate are respectively determined as

\[ N = \sqrt{\frac{72}{\tau_1}} \text{ and } \alpha = \sqrt{r}. \]

**Proof.** We have

\[ V(k+1) \leq r^{-1}V(k) \leq ... \leq r^{-k-1}V(0). \]

Thus

\[ V(k) \leq r^{-k} \phi \in \mathbb{Z}^+. \]

By taking into account, we obtain

\[ \|x(k, \phi)\| \leq \sqrt{\frac{72}{\tau_1}} \|\phi\| \alpha^{-k}, \quad k \in \mathbb{Z}^+ \]

where \( \alpha = \sqrt{r}. \) This completes the proof.

**III. MAIN RESULTS**

The following notations are specifically used in this development. For given integers \( \tau_m, \tau_M \) satisfying \( 0 < \tau_m < \tau_M \) and any integer number \( \delta \in (0, \tau_M - \tau_m) \), symmetric positive definite matrices \( P, Q_j, R_j, j = 1, 2, 3, S_1, S_2 \) and a matrix \( X \), \( X' \) such as the following constants.

Vectors

\[ \xi(k) = \left[ \begin{array}{c} x_T(k) \nabla_T(k - \tau_m) x_T(k + \tau_m - \tau) \nabla_T(k - \tau) \end{array} \right]^T, \]

\[ \omega(k) = \left[ \begin{array}{c} x_T(k) \nabla_T(k) \end{array} \right]^T, \]

\[ \Phi(k) = \left[ \begin{array}{c} x_T(k) \nabla_T(k + 1) \nabla_T(k + \tau_m - \tau) \nabla_T(k + \tau_m + \tau) \end{array} \right]^T. \]

And matrices

\[ \Phi = r m \Phi_1 + (\tau - \tau_m) \Phi_2 + (\tau_m - \tau) \Phi_3, \]

\[ \Pi_1 = \left[ \begin{array}{c} e_1 & e_2 & e_3 \end{array} \right]^T, \]

\[ \Pi_2 = \left[ \begin{array}{c} e_4 & e_5 & e_6 \end{array} \right]^T, \]

\[ \Pi_3 = \left[ \begin{array}{c} e_7 & e_8 & e_9 \end{array} \right]^T, \]

\[ \Phi_1 = \left[ \begin{array}{c} e_1 & e_2 & e_3 \end{array} \right]^T, \]

\[ \Phi_2 = \left[ \begin{array}{c} e_4 & e_5 & e_6 \end{array} \right]^T, \]

\[ \Phi_3 = \left[ \begin{array}{c} e_7 & e_8 & e_9 \end{array} \right]^T. \]

**Theorem:** For given integers \( 0 < \tau_m < \tau_M \), if there exist a scalar \( r > 1 \), an integer \( \delta \in (0, \tau_M - \tau_m) \), symmetric positive definite matrices \( P, Q_j, R_j, j = 1, 2, 3, S_1, S_2 \) and a matrix \( X, X' \) such that the following inequalities hold

\[ Q_0 - Q_1 < 0, \]

\[ Q_2 - Q_0 < 0. \]

Then system is exponentially stable with the exponential decay rate \( \alpha = \sqrt{r} \).

Moreover, every solutions of system satisfies

\[ \|x(k, \phi)\| \leq \sqrt{\frac{72}{\tau_1}} \|\phi\| \alpha^{-k}, \quad k \in \mathbb{Z}^+ \]

Where

\[ \gamma_1 = \lambda_{\omega} (P), \]

\[ \gamma_2 = (1 + \gamma_1) \lambda_{\omega} (P) \]

\[ + \frac{1}{1 - r^{-\gamma_3}} \lambda_{\omega} (Q_1) + \frac{1 - r^{-\gamma_3}}{1 - r^{-\gamma_3}} \lambda_{\omega} (Q_2) + \frac{1 - r^{-\gamma_3}}{1 - r^{-\gamma_3}} \lambda_{\omega} (Q_3) + \frac{1 - r^{-\gamma_3}}{1 - r^{-\gamma_3}} \lambda_{\omega} (R) \]

\[ + \frac{1}{1 - r^{-\gamma_3}} \lambda_{\omega} (S_1) + \frac{1 - r^{-\gamma_3}}{1 - r^{-\gamma_3}} \lambda_{\omega} (S_2). \]

**Proof.** Define \( y(k) = x(k + 1) - x(k) = (A - I)x(k) + Adx(k - \tau(k)). \) Now, for the sake of getting more information on system the interval time-varying delay \( [\tau_m, \tau_M] \) is divided into two nonuniform subintervals. Note that, for any \( k \in \mathbb{Z} \), we have either \( \tau(k) \in [\tau_m, \tau_M] \) or \( \tau(k) \in (\tau_m, \tau_M) \). Let us define two sets

\[ \Gamma_1 = \{ k \in \mathbb{Z} | \tau(k) \in [\tau_m, \tau_M] \}, \]

\[ \Gamma_2 = \{ k \in \mathbb{Z} | \tau(k) \in (\tau_m, \tau_M) \}. \]

Consider the following Lyapunov-Krasovskii functional

\[ V(k) = V_1(k) + V_2(k) + V_3(k) + V_4(k) \]

Where

\[ V_1(k) = \zeta^T(k)P\zeta(k), \]

\[ V_2(k) = \sum_{s=k-\tau_m}^{k-1} r^{-s+1} x^T(s)Q_1 x(s) + \sum_{s=k-\tau}^{k-\tau_m-1} r^{-s+1} x^T(s)Q_2 x(s) + \sum_{s=k-\tau}^{k-1} r^{-s+1} x^T(s)Q_3 x(s) \]

\[ + \sum_{s=k-\tau}^{k-1} r^{-s+1} x^T(s)Q_4 x(s), \]

\[ V_3(k) = -\sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ - \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ + \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s), \]

\[ V_4(k) = \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s). \]

By taking the forward difference of \( V_1(k) \) along the solutions of system, we have

\[ \Delta V_1(k) = V_1(k + 1) - V_1(k) = r^{-1} \zeta^T(k) P \zeta(k) \]

\[ + \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s), \]

\[ \text{Therefore, } \Delta V_1(k) \text{ can be obtained of the form} \]

\[ \Delta V_1(k) = \zeta^T(k)(I - 1)P\zeta(k) + (r^{-1} - 1)V_1(k), \]

Where

\[ \zeta(k + 1) = \begin{bmatrix} x(k + 1) \\ \sum_{s=k-1}^{k} x(s) \end{bmatrix} = \begin{bmatrix} x(k) - x(k - \tau_m) + \sum_{s=k-1}^{k} x(s) \\ x(k - \tau_m) - x(k - \tau) + \sum_{s=k-1}^{k-\tau_m} x(s) \end{bmatrix} \]

\[ \text{Therefore, } \Delta V_1(k) \text{ can be obtained of the form} \]

\[ \Delta V_1(k) = \zeta^T(k)(I - 1)P\zeta(k) + (r^{-1} - 1)V_1(k), \]

The forward differences of \( V_2(k) \) and \( V_3(k) \) are obtained as

\[ \Delta V_2(k) = x^T(k)Q_1 x(k) + r^{-1} [P_{11} x(k - \tau_m) + P_{12} x(k - \tau_m) + P_{13} x(k - \tau_m) + \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ \text{The forward differences of } V_2(k) \text{ and } V_3(k) \text{ are obtained as} \]

\[ \Delta V_2(k) = x^T(k)Q_1 x(k) + r^{-1} \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s), \]

\[ \Delta V_3(k) = r^s\rho^T(s)R_3 \rho(s), \]

\[ \text{Similarly, the difference of } V_4(k) \text{ along solutions} \]

\[ \text{of system is calculated as follows.} \]

\[ \Delta V_4 = \rho^T(s)R_3 \rho(s), \]

\[ \text{Similarly, we also have} \]

\[ \Delta V_1(k) = \zeta^T(k)(I - 1)P\zeta(k) + \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s), \]

\[ \text{By using} \]

\[ \text{Lemma 1, the following inequality is obtained as} \]

\[ - \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ \leq \eta_1^T R_3 \eta_1 \]

\[ \text{From Lemma 2, we have} \]

\[ \text{and} \]

\[ \text{Similarly, for the different } \Delta V_4(k), \text{ by using Lemma 1, the following inequalities are obtained as} \]

\[ - \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ \leq -r_3 \zeta^T(k)P_1 \zeta(k). \]

\[ \text{Similarly, for the different } \Delta V_4(k), \text{ by using Lemma 1, the following inequalities are obtained as} \]

\[ - \sum_{s=k-\tau}^{k-1} r^s\rho^T(s)R_3 \rho(s) \]

\[ \leq -r_3 \zeta^T(k)P_1 \zeta(k). \]
\[
\Delta V (k) + (1 - r^{-1})V (k) \leq \xi^T (k)(\Omega 0 - \Omega 2)\xi (k), \forall k \in \mathbb{Z}^+.
\]

We now obtain
\[
\Delta V (k) + (1 - r^{-1})V (k) \leq \xi^T (k)(\Omega 0 - \Omega 1)\xi (k), \forall k \in \mathbb{Z}^+.
\]

Remark 1 By comparison to other LKFs existing in the literature, the new LKFs proposed in this paper contain a new augmented vector \( \zeta (k) \) in \( V_1 (k) \) and two triple summation terms in \( V_4 (k) \). As a result, the information about the current values of the state variables \( x(k) \) and their history are exploited, leading to less conservative stability conditions.

Remark 2 It should be noted that the stability conditions, established in Theorem 20, contain the tuning parameters \( \alpha \) and \( \delta \) so there remains the interesting question as how to find the optimal combination of these parameters. A direct method to solve that problem is to choose a cost function \( t_{\text{min}} \) that is obtained while solving the feasibility problem using Matlab’s LMI toolbox. When \( t_{\text{min}} \) is positive, the combination of the tuning parameters \( \alpha \) and \( \delta \) does not allow a feasible solution to the set of LMIs \([15]\). By using a numerical optimisation algorithm, such as the program \( \text{fminsearch} \) in the optimisation toolbox of Matlab, we can find the solution of the cost function \( t_{\text{min}} \). The optimal combination of two parameters is obtained when the minimum value of the cost function is negative \([16]\).

Remark 3 The convergence rate of the system can be directly chosen from this proposed approach. Moreover, the exponential stability conditions given in Theorem 20 are derived in terms of LMIs without introducing any free-weighting matrices. Therefore, the obtained stability conditions may involve fewer decision variables, and hence reduce the computational complexity.

IV. Numerical examples

For illustration of the effectiveness of the proposed approach in relaxing the conservatism of the stability conditions, let us consider the system, given in \([7, 11, 14, 10] \) as follows
\[
x(k + 1) = \begin{pmatrix}
0.8 & 0 \\
0.05 & 0.9
\end{pmatrix} x(k) + \begin{pmatrix}
-0.1 & 0 \\
-0.2 & -0.1
\end{pmatrix} x(k - \tau (k))
\]

For this system, the asymptotic stability conditions proposed in the aforementioned studies gave the allowable upper bounds \( \tau M \) with various values of \( \tau m \) as listed in Table 3.1 below. Here, from Theorem 20 and Remark 2, we find the optimal value for \( \tau \) as \( r \)
As a result, the exponential convergence rate is calculated as $\alpha = 1.0005$ and the upper bounds $\tau_M$ are also given correspondingly in Table 3.1. It can be seen that all the upper bounds $\tau_M$ obtained by using Theorem 20 are larger than those obtained in the papers mentioned above. As explained in Remark 1, this also confirms the improvement of this approach on stability conditions as compared to existing results.

| MABs of $\tau_M$ for different values of $\tau_M$ |
|--------|-------|-------|-------|-------|-------|
| $\tau_M$ | 1     | 3     | 5     | 7     | 11    |
| Gao et al. (2007) | 12    | 13    | 13    | 14    | 16    |
| Zhang et al. (2008) | 12    | 13    | 14    | 15    | 17    |
| He et al. (2008) | 17    | 17    | 18    | 18    | 20    |
| Meng et al. (2010) | 17    | 17    | 18    | 20    | 22    |
| Li and Gao (2011) | -     | 18    | 19    | 21    | 25    |
| Kao (2012) | 17    | 18    | 19    | 21    | 25    |
| This paper | 19    | 19    | 20    | 21    | 22    |

has been proposed. By combining the reciprocally convex approach with the delay decomposition technique, improved exponential stability conditions are derived in terms of LMIs. In comparison with existing results in the literature, the obtained conditions are less conservative, judging by a larger maximum allowable bound. The effectiveness of the proposed approach is illustrated through numerical examples.

VI. REFERENCES

